On isometries of Product of normed linear spaces

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Abstract. We give a condition on norms under which two vector normed spaces X and Y are isometrically isomorphic if and only if $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are isometrically isomorphic. We also prove that this result fail for arbitrary norms even if $X = Y = \mathbb{R}^2$ by building a generic counterexamples.

Keyword, phrase: Normed vector space and isometries.

1 Introduction

We are interested in this paper in the following question. Let X and Y be vector spaces and let N_X and N_Y be two norms on $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ respectively. The norm N_X on X (and in a similar way N_Y) denotes $N_X(x, 0)$ for all $x \in X$.

Problem. It is true that $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ are isometrically isomorphic if and only if (X, N_X) and (Y, N_Y) are isometrically isomorphic?

We begin by showing that in the general case the answer to this question is no for arbitrary norms N_X and N_Y, even when X and Y are two dimensional vector spaces, by constructing a generic counterexamples (See Theorem 1). We prove then in Theorem 2 that the result is true for all norms (N_X, N_Y) satisfying the following property (P).

Definition 1 *Let X and Y be two vector spaces. Let N_X and N_Y be two norms on $X \times \mathbb{R}$ and $X \times \mathbb{R}$ respectively. We say the the pair (N_X, N_Y) satisfy the property (P) if for all $x \in X$ and all $y \in Y$:*

$$N_X(x, 0) = N_Y(y, 0) \Rightarrow N_X(x, \lambda) = N_Y(y, \lambda), \forall \lambda \in \mathbb{R}.$$

In all the article we identify X with $X \times \{0\}$ and the norm N_X on X denotes $N_X(x, 0)$ for all $x \in X$.

Exemples 1 *Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed vector spaces. Let $p \in [1, +\infty[$ and*

$$N_{X,p}(x, t) := (\|x\|_X^p + |t|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x, t) := \max(\|x\|_X, |t|),$$

for all $(x, t) \in X \times \mathbb{R}$. In a similar way we define $N_{Y,p}$ and $N_{Y,\infty}$. Then the pairs $(N_{X,p}, N_{Y,p})$ and $(N_{X,\infty}, N_{Y,\infty})$ satisfies the property (P).

We give in the following proposition a more general examples.

Proposition 1 *Let $N_{\mathbb{R}^2}$ be any norm on $\mathbb{R}^2$ such that $N_X(x,t) := N_{\mathbb{R}^2}(\|x\|_X, |t|)$ for all $(x,t) \in X \times \mathbb{R}$ defined a norm on $X \times \mathbb{R}$ (Similarly we define N_Y on $Y \times \mathbb{R}$). Then (N_X, N_Y) satisfy the property (P).*

Proof. Let $x \in X$ and $y \in Y$ be such that $N_X(x,0) = N_Y(y,0)$. Then $N_{\mathbb{R}^2}(\|x\|_X, 0) = N_{\mathbb{R}^2}(\|y\|_Y, 0)$ and so $\|x\|_X N_{\mathbb{R}^2}(1,0) = \|y\|_Y N_{\mathbb{R}^2}(1,0)$, which implies that $\|x\|_X = \|y\|_Y$. It follows that $N_{\mathbb{R}^2}(\|x\|_X, |\lambda|) = N_{\mathbb{R}^2}(\|y\|_Y, |\lambda|)$ for all $\lambda \in \mathbb{R}$. In other words $N_X(x,\lambda) = N_Y(y,\lambda)$ for all $\lambda \in \mathbb{R}$.$\blacksquare$

The problem mentioned above was motivated at the first time in [1] by questions connected to the Banach-Stone theorem, and solved positively only for the particular norms $N_{X,p}$ and $N_{Y,p}$ when $p \in [1, +\infty[\setminus \{2\}$. The technique used in [1] did not include the case p=2. The property (P) hear is more general and allowed to include varied norms. We give in section4. other simple examples of applications of Theorem 2.

2 A generic counterexample.

Theorem 1 *Let $X = Y = \mathbb{R}^2$. For each norm $\|.\|_X$ on X there exists a norm $\|.\|_Y$ on Y, a norm N_X on $X \times \mathbb{R}$ and a norm N_Y on $Y \times \mathbb{R}$ such that :*

(1) $(X, \|.\|_X)$ *is not isometrically isomorphic to* $(Y, \|.\|_Y)$.

(2) $(X \times \mathbb{R}, N_X)$ *is isometrically isomorphic to* $(Y \times \mathbb{R}, N_Y)$.

(3) *the restriction of N_X to X coincide with $\|.\|_X$ and the restriction of N_Y to Y coincide with $\|.\|_Y$.*

Proof. Let $p \in [1, +\infty[$. Let us define N_X and N_Y as follow :

$$N_X(x_1, x_2, t) := (\|(x_1, x_2)\|_X^p + |t|^p)^{\frac{1}{p}}, \quad \forall (x_1, x_2, t) \in X \times \mathbb{R}$$

and

$$N_Y(y_1, y_2, s) := (|y_2|^p + \frac{\|(y_1, s)\|_X^p}{a^p})^{\frac{1}{p}}, \quad \forall (y_1, y_2, s) \in Y \times \mathbb{R}.$$

Where $a = \|(1,0)\|_X$. Let us define the norm $\|.\|_{Y,p}$ on Y as follows $\|(y_1, y_2)\|_{Y,p} := (|y_1|^p + |y_2|^p)^{\frac{1}{p}}$ for all $(y_1, y_2) \in Y$. Clearly,

$$N_X(x_1, x_2, 0) = \|(x_1, x_2)\|_X, \quad \forall (x_1, x_2) \in X$$

and

$$N_Y(y_1, y_2, 0) = (|y_1|^p + |y_2|^p)^{\frac{1}{p}} := \|(y_1, y_2)\|_{Y,p}, \quad \forall (y_1, y_2) \in Y.$$

(Since $\frac{\|(y_1,0)\|_X^p}{a^p} = |y_1| \frac{\|(1,0)\|_X^p}{a^p} = |y_1|$). On the other hand, the following map is an isometric isomorphism:

$$\begin{array}{rcl} \Theta : (X \times \mathbb{R}, N_X) & \to & (Y \times \mathbb{R}, N_Y) \\ (x_1, x_2, t) & \mapsto & (ax_1, t, ax_2). \end{array}$$

Now, there exist to cases:

Case 1 : If every point of the sphere S_X of X is an extreme point, we choose $p = 1$ and so S_Y has a non extreme point since in this case $\|(y_1, y_2)\|_{Y,1} = |y_1| + |y_2|$ (For example $(\frac{1}{2}, \frac{1}{2})$ is not extreme for $\|.\|_{Y,1}$). Consequently X and Y cannot be isometrically isomorphic.

Case 2 : If there exists some point of the sphere S_X which is not extreme point then we choose $p = 2$ and so every points of S_Y is an extreme point since $\|(y_1, y_2)\|_{Y,2} = (|y_1|^2 + |y_2|^2)^{\frac{1}{2}}$ is the euclidean norm. Also X and Y cannot be isometrically isomorphic.$\blacksquare$

3 Isometries between product spaces.

Theorem 2 *Let X and Y be a vector spaces. Suppose that (N_X, N_Y) satisfy the property (P). Then $(X \times \mathbb{R}, N_X)$ and $(Y \times \mathbb{R}, N_Y)$ are isometrically isomorphic if and only if (X, N_X) and (Y, N_Y) are isometrically isomorphic.*

The proof of the above theorem is given in section 3.2 after some lemmas.

3.1 Notations and lemmas.

We need some notations and lemmas. Let $\Theta : (X \times \mathbb{R}, N_X) \to (Y \times \mathbb{R}, N_Y)$ be an isomorphism isometric. We set $(a, u) = \Theta^{-1}(0, 1)$ and $(b, v) = \Theta(0, 1)$. Let us define the linear continuous map χ_X as follow :

$$\begin{array}{rcl} \chi_X : X \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (x, t) & \mapsto & t \end{array}$$

We define analogously the map χ_Y by

$$\begin{array}{rcl} \chi_Y : Y \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (y, t) & \mapsto & t \end{array}$$

We obtain the following linear map on $Y \times \{0\}$:

$$\chi_X \circ \Theta^{-1} : Y \times \{0\} \longrightarrow \mathbb{R}$$

Analogously we have also the linear map on $X \times \{0\}$:

$$\chi_Y \circ \Theta : X \times \{0\} \longrightarrow \mathbb{R}$$

Let us set $X_0 := Ker\,(\chi_Y \circ \Theta)$ and $Y_0 := Ker\,\left(\chi_X \circ \Theta^{-1}\right)$.

Remark 1 *The linear spaces X_0 and Y_0 are not necessarily closed since χ_X and χ_Y are not necessarily continuous.*

Lemma 1 *X_0 and Y_0 are isometrically isomorphic. More precisely, the map*

$$\begin{array}{rcl} \Theta : (X_0, N_X) & \to & (Y_0, N_Y) \\ (z, 0) & \mapsto & \Theta(z, 0) \end{array} \tag{1}$$

is an isomorphism isometric.

Proof. Since Θ is an isomorphism isometric, it suffices to show that the restriction of Θ to X_0 is onto. Indeed, let $(y, 0) \in Y_0$. Clearly, $(z, 0) := \Theta^{-1}(y, 0) \in X_0$ since $\chi_Y \circ \Theta(\Theta^{-1}(y, 0)) = \chi_Y(y, 0) = 0$ and we have $(y, 0) = \Theta(z, 0)$.$\blacksquare$

Lemma 2 *We have only two cases.*

Case1: $u \neq 0$. *In this case, we have $X \times \{0\} = X_0$.*

Case2: $u = 0$. *In this case we have $\Theta^{-1}(0, 1) = (a, 0)$ and $X \times \{0\} = X_0 \oplus \mathbb{R}(a, 0)$. Similarly we have,*

Case1: $v \neq 0$. *In this case, we have $Y \times \{0\} = Y_0$.*

Case2: $v = 0$. *In this case we have $\Theta(0, 1) = (b, 0)$ and $Y \times \{0\} = Y_0 \oplus \mathbb{R}(b, 0)$.*

Proof. For all $x \in X$ there exists $(y_x, \lambda_x) \in Y \times \mathbb{R}$ such that

$$\begin{aligned}(x,0) = \Theta^{-1}(y_x,\lambda_x) &= \Theta^{-1}(y_x,0) + \lambda\Theta^{-1}(0,1)\\ &= \Theta^{-1}(y_x,0) + \lambda_x(a,u)\\ &= \Theta^{-1}(y_x,0) + (\lambda_x a, \lambda_x u)\end{aligned} \tag{2}$$

Since $\Theta^{-1}(y_x,0) \in X_0 \subset X \times \{0\}$ and also $(x,0) \in X \times \{0\}$, then from the above equation we obtain that $(\lambda_x a, \lambda_x u) \in X \times \{0\}$ which implies that $\lambda_x u = 0$. So we have :

Case1: $u \neq 0$. In this case, $X \times \{0\} = X_0$. Indeed, if $u \neq 0$ then $\lambda_x = 0$ and so $(x,0) = \Theta^{-1}(y_x,0) \in X_0$, for all $x \in X$ i.e $X \times \{0\} \subset X_0$. On the other hand we know that $X_0 \subset X \times \{0\}$.

Case2: $u = 0$. In this case we have $\Theta^{-1}(0,1) = (a,0)$ and so $X = X_0 \oplus \mathbb{R}(a,0)$. Indeed, We have $X_0 \cap \mathbb{R}(a,0) = (0,0)$, since if α is a real number such that $\alpha(a,0) \in X_0$ then $0 = \chi_Y \circ \Theta\left(\alpha(a,0)\right) = \alpha\chi_Y(0,1) = \alpha$. In other words from (2), for all $x \in X$, there exist $(y_x,\lambda_x) \in Y \times \mathbb{R}$ such that

$$(x,0) \quad = \quad \Theta^{-1}(y_x,0) + \lambda_x(a,0).$$

whith $\Theta^{-1}(y_x,0) \in X_0$. Thus $X \times \{0\} \subset X_0 \oplus \mathbb{R}(a,0) \subset X \times \{0\}$ and so $X \times \{0\} = X_0 \oplus \mathbb{R}(a,0)$.

In a similar way we obtain the second part of the lemma.■

Lemma 3 *We have, $u = 0$ if and only if $v = 0$.*

Proof. Suppose that $v = 0$. Then for all $(x,t) \in X \times \mathbb{R}$, we have $\Theta(x,t) = \Theta(x,0) + \Theta(0,t) = \Theta(x,0) + t\Theta(0,1) = \Theta(x,0) + t(b,v) = \Theta(x,0) + (tb,0)$. Now, we are going to prove that $u = 0$. Suppose that the contrary hold, that is $u \neq 0$. Then $X \times \{0\} = X_0$ (See the case 1. in Lemma 2). So $\Theta(x,0) \in \Theta(X \times \{0\}) = \Theta(X_0) = Y_0$, since Θ is an isomorphism isometric from X_0 onto Y_0 (See the formula (1)). Now since $Y_0 \subset Y \times \{0\}$, then $\Theta(x,0) + t(b,0) \in Y \times \{0\}$. In other words, $\Theta(x,t) \in Y \times \{0\}$ for all $(x,t) \in X \times \mathbb{R}$. So $\Theta(X \times \mathbb{R}) \subset Y \times \{0\}$. But Θ is an isomorphism between $X \times \mathbb{R}$ and $Y \times \mathbb{R}$. This implies that $Y \times \{0\} = Y \times \mathbb{R}$ which is impossible. Thus $u = 0$. In a similar way we obtain the converse.■

3.2 Proof of Theorem 2 and some corollaries.

We give now the proof of the main result.

Proof of Theorem 2. For the "if" part, let $T : (X, N_X) \to (Y, N_Y)$ be an isomorphism isometric. Let us define $\Theta : (X \times \mathbb{R}, N_X) \to (Y \times \mathbb{R}, N_Y)$ by $\Theta(x,\lambda) = (T(x),\lambda)$. Then, clearly Θ is an isomorphism and by the property (P) it is also isometric. We prove now the "only if part". By combining Lemma 2 and Lemma 3 we have that:

Case1. If $u \neq 0$ and $v \neq 0$, then $X \times \{0\} = X_0$ and $Y \times \{0\} = Y_0$. So by Lemma 1 we conclude that $X \times \{0\}$ and $Y \times \{0\}$ are isometrically isomorphic for the norms N_X and N_Y. So (X, N_X) and (Y, N_Y) are isometrically isomorphic.

Case2. If $u = 0$ and $v = 0$, using Lemma 2 we have that $\Theta^{-1}(0,1) = (a,0)$ and $X \times \{0\} = X_0 \oplus \mathbb{R}(a,0)$ and $\Theta(0,1) = (b,0)$ and $Y \times \{0\} = Y_0 \oplus \mathbb{R}(b,0)$. Now we prove that the map

$$\begin{aligned}\psi : X \times \{0\} = X_0 \oplus \mathbb{R}(a,0) &\to Y \times \{0\} = Y_0 \oplus \mathbb{R}(b,0)\\ (z,0) + \lambda(a,0) &\mapsto \Theta(z,0) + \lambda(b,0)\end{aligned}$$

is an isomorphism isometric. Indeed, the fact that ψ is linear and onto map is clear by using Lemma 1. Let us prove that ψ is isometric for the norms N_X and N_Y. Since $(z,0) \in X_0$, by (1) there exist $(y,0) \in Y_0$ such that $\Theta(z,0) = (y,0)$. Since

$$\begin{aligned}(z,0) + \lambda(a,0) &= \Theta^{-1}(\Theta(z,0)) + \lambda\Theta^{-1}(0,1)\\ &= \Theta^{-1}\left(\Theta(z,0) + (0,\lambda)\right)\\ &= \Theta^{-1}(y,\lambda)\end{aligned}$$

then, using the fact that Θ^{-1} is isometric we have

$$\begin{aligned} N_X((z,0)+\lambda(a,0)) &= N_X(\Theta^{-1}(y,\lambda)) \\ &= N_Y(y,\lambda). \end{aligned} \tag{3}$$

On the other hand we known that $(b,0)=\Theta(0,1)$ so $\Theta(z,0)+\lambda(b,0)=\Theta(z,0)+\lambda\Theta(0,1)=\Theta(z,\lambda)$. Thus, using the fact that Θ is isometric we have,

$$\begin{aligned} N_Y(\psi\left((z,0)+\lambda(a,0)\right)) &= N_Y(\Theta(z,0)+\lambda(b,0)) \\ &= N_Y(\Theta(z,\lambda)) \\ &= N_X(z,\lambda). \end{aligned} \tag{4}$$

But $N_X(z,0)=N_Y(y,0)$ since $\Theta(z,0)=(y,0)$ and Θ is isometric. Since (N_X,N_Y) satisfy the property (P) then $N_X(z,\lambda)=N_Y(y,\lambda)$. Thus, using the formulas (3) and (4) we obtain that ψ is isometric.$\blacksquare$

Remark 2 *By induction, we can easily extend the above theorem to $X\times\mathbb{R}^n$ $(n\in N^*)$ if we assume that (N_X,N_Y) is a pair of norms satisfying the following property (P^n): for all $x\in X$ all $y\in Y$, all $i\in\{1,2,...n\}$ and all $(s_1,s_2,...,s_i);(s'_1,s'_2,...,s'_i)\in\mathbb{R}^i$:*
if $N_X(x,s_1,s_2,...,s_i,0,0,...,0)=N_Y(y,s'_1,s'_2,...,s'_i,0,0,...,0)$ then
$N_X(x,s_1,s_2,...,s_i,\lambda,0,...,0)=N_Y(y,s'_1,s'_2,...,s'_i,\lambda,0,...,0),\forall\lambda\in\mathbb{R}$.

Exemples 2 *Let $p\in[1,+\infty[$, and*

$$N_{X,p}(x,s_1,...,s_n)=(\|x\|_X^p+\sum_{k=1}^n|s_k|^p)^{\frac{1}{p}},$$

$$N_{X,\infty}(x,s_1,...,s_n)=\max(\|x\|_X,|s_1|,...,|s_n|)$$

for all $(x,s_1,...,s_n)\in X\times\mathbb{R}^n$. In a similar way we define $N_{Y,p}$ and $N_{Y,\infty}$. Then the pairs $(N_{X,p},N_{Y,p})$ and $(N_{X,\infty},N_{Y,\infty})$ satisfies the property (P^n).

Corollary 1 *Let X and Y be a vector spaces. Let $n\in\mathbb{N}^*$ and suppose that (N_X,N_Y) satisfy (P^n). Then $(X\times\mathbb{R}^n,N_X)$ and $(Y\times\mathbb{R}^n,N_Y)$ are isometrically isomorphic if and only if (X,N_X) and (Y,N_Y) are isometrically isomorphic.*

As a remark we have the following corollary for inner product spaces. Note that a non complete inner product space has no orthonormal basis in general (See [5]). The symbol $\cong$ means "isometrically isomorphic".

Corollary 2 *Let $(H,\|.\|_H)$ and $(L,\|.\|_L)$ be two inner product space (not necessary complete). Then $(H,\|.\|_H)\cong(L,\|.\|_L)$ if and only if for all finite dimensional subspaces $E\subset H$ and $F\subset L$ such that $dim(E)=dim(F)$ we have that $(E^\perp,\|.\|_H)\cong(F^\perp,\|.\|_L)$. Where $E^\perp$ and $F^\perp$ denotes the orthogonal of E and F respectively.*

Proof. Let $E\subset H$ and $F\subset L$ such that $dim(E)=dim(F)=n$ for $n\in\mathbb{N}$. By the classical projection theorem on a complete vector subspace of an inner product space, we have $H=E^\perp\oplus E$ and $L=F^\perp\oplus F$. On the other hand it is clear that $(H,\|.\|_H)\cong(E^\perp\times\mathbb{R}^n,N_{E^\perp,2})$ and $(L,\|.\|_L)\cong(F^\perp\times\mathbb{R}^n,N_{F^\perp,2})$, where $N_{E^\perp,2}$ and $N_{F^\perp,2}$ are defined as in the Example 2 with $p=2$. Since $(N_{E^\perp,2},N_{F^\perp,2})$ satisfy (P^n) then from Corollary 1 we obtain $(E^\perp,\|.\|_H)\cong(F^\perp,\|.\|_L)$. The converse is clear.$\blacksquare$

4 Applications.

We give in this section two applications of Theorem 2. We denote by $(C^1[0,1], N_{C^1[0,1]})$ the space of continuously differentiable functions on $[0,1]$ endowed with the norm $N_{C^1[0,1]}(f) := N_{\mathbb{R}^2}(\|f'\|_\infty, |f(0)|)$, where $N_{\mathbb{R}^2}$ denotes any norm satisfying Proposition 1. Let $(X, \|.\|_X)$ be a Banach space. We denote by N_X the norm defined on $X \times \mathbb{R}$ by $N_X(x,t) := N_{\mathbb{R}^2}(\|x\|_X, |t|)$ for all $(x,t) \in X \times \mathbb{R}$. Finally, we denote by $(C[0,1], \|.\|_\infty)$ the space of continuous functions on $[0,1]$ endowed with the supremum norm.

Proposition 2 *We have $(X \times \mathbb{R}, N_X) \cong (C^1[0,1], N_{C^1[0,1]})$, if and only if $(X, \|.\|_X) \cong (C[0,1], \|.\|_\infty)$.*

Proof. Let us define the norm $N_{C[0,1]}$ on $C[0,1] \times \mathbb{R}$ by $N_{C[0,1]}(g,t) := N_{\mathbb{R}^2}(\|g\|_\infty, |t|)$ for all $(g,t) \in C[0,1] \times \mathbb{R}$. Let us consider the map

$$\begin{array}{rcl} \chi : (C^1[0,1], N_{C^1[0,1]}) & \to & (C[0,1] \times \mathbb{R}, N_{C[0,1]}) \\ f & \mapsto & (f', f(0)) \end{array}$$

Clearly, χ is an isomorphism isometric. So we have $(X \times \mathbb{R}, N_X) \cong (C[0,1] \times \mathbb{R}, N_{C[0,1]})$. Since $(N_X, N_{C[0,1]})$ satisfy the property (P) by Proposition 1 then using Theorem 2 we obtain that $(X, \|.\|_X) \cong (C[0,1], \|.\|_\infty)$, since $N_X(\|x\|_X, 0) = \|x\|_X N_{\mathbb{R}^2}(1,0)$ and $N_{C[0,1]}(g,0) = \|g\|_\infty N_{\mathbb{R}^2}(1,0)$.$\blacksquare$

Let us recall some notions. Let K and C be convex subsets of vector spaces. A function $T : K \to C$ is said to be affine if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$. The set of all continuous real-valued affine functions on a convex subset K of a topological vector space will be denoted by $Aff(K)$. Clearly, all translates of continuous linear functionals are elements of *Aff*(K), but the converse in not true in general (see [4] page 22.). However, we do have the following relationship.

Proposition 3 *([4], Proposition 4.5) Assume that K is a compact convex subset of a separated locally convex space X then*

$$\left\{a \in Aff(K) : a = r + x^*_{|K} \text{ for some } x^* \in X^* \text{ and some } r \in R\right\}$$

is dense in $(Aff(K), \|.\|_\infty)$, where $\|.\|_\infty$ denotes the norm of uniform convergence.

But in the particular case when X is a Banach space and $K = (B_{X^*}, w^*)$ is the unit ball of the dual space X^* endowed with the weak star topology, the well known result due to Banach and Dieudonné states that:

Theorem 3 (Banach-Dieudonné). *The space $(Aff_0(B_{X^*}), \|.\|_\infty)$ is isometrically identified to $(X, \|.\|)$. In other words, $Aff_0(B_{X^*}) = \left\{\hat{z}_{|B_{X^*}} : z \in X\right\}$. Where $Aff_0(B_{X^*})$ denotes the space of all affine weak star continuous functions that vanish at 0 and $\hat{z} : p \mapsto p(z)$ for all $p \in X^*$ and $\hat{z}_{|B_{X^*}}$ denotes the restriction of $\hat{z}$ to B_{X^*}.*

Now, let X and Y be two Banach spaces and let us endowed the space $Aff(B_{X^*})$ (and in a similar way the space $Aff(B_{Y^*})$) with the norm $N(f) := N_{\mathbb{R}^2}(\|f - f(0)\|_\infty, |f(0)|)$ for all $f \in Aff(B_{X^*})$, where $N_{\mathbb{R}^2}$ denotes any norm on $\mathbb{R}^2$ satisfying Proposition 1. We obtain the following version of the Banach-Stone theorem for affine functions (For more information about the Banach-Stone theorem see [2] and [3]).

Proposition 4 *Let X and Y be two Banach spaces. Then the following assertions are equivalent.*

(1) *$(Aff(B_{X^*}), N)$ and $(Aff(B_{Y^*}), N)$ are isometrically isomorphic.*

(2) *$(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are isometrically isomorphic.*

Proof. Let $\tilde{N}$ be the norm on $Aff_0(B_{X^*}) \times \mathbb{R}$ defined by $\tilde{N}(f_0, t) := N_{\mathbb{R}^2}(\|f_0\|_\infty, |t|)$ for all $(f_0, t) \in Aff_0(B_{X^*}) \times \mathbb{R}$. Let us consider the map,

$$\begin{array}{rcl} \chi : (Aff(B_{X^*}), N) & \rightarrow & (Aff_0(B_{X^*}) \times \mathbb{R}, \tilde{N}) \\ f & \mapsto & (f - f(0), f(0)) \end{array}$$

Clearly, χ is an isometric isomorphism. Thus using Theorem 1 we have that $(Aff(B_{X^*}), N)$ and $(Aff(B_{X^*}), N)$ are isometrically isomorphic if and only if $(Aff_0(B_{X^*}), \|.\|_\infty)$ and $(Aff_0(B_{Y^*}), \|.\|_\infty)$ are isometrically isomorphic, which is equivalent by Theorem 3 to the fact that $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are isometrically isomorphic.■

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